J. Fluid Mech. (1991), *vol.* 226, *pp.* 175–187
Printed in Great Britain

Flow around an unsteady thin wing close to curved ground

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(Received 22 December 1989 and in revised form 10 September 1990)

The method of matched asymptotic expansions is applied to the flow analysis of a three-dimensional thin wing, moving uniformly in very close proximity to a curved ground surface. Four flow regions, i.e. exterior, bow, gap, and wake, are analysed and matched in an appropriate sequence. The solutions in expansions up to third order are given both in nonlinear and linear cases. It is shown that the flow above the wing is reduced to a direct problem, and the flow beneath it appears to be a two-dimensional channel flow. The wake assumes a vortex-sheet structure close to the curved ground, undulating with the amplitude of the ground curvature, and the flow beneath it is also two-dimensional channel flow. As a consequence, an equivalence is found between the extreme curved-ground effect and the corresponding flat-ground effect, which can be treated by the image method.

1. Introduction

Studies of the aerodynamic ground effect for moving vehicles have attracted many investigators in the past half-century. There is a considerable literature dealing with the ordinary flat-ground effect.

The present study is focused on the so-called extreme ground effect for a body moving in very close proximity to a weakly curved ground (or water surface). It is of practical importance in the context of high-speed ground transportation vehicles, and with respect to the interaction between a ship hull and an adjacent canal wall or second ship (Tuck 1975). It is also of interest with respect to wing-in-ground-effect vehicles, which are promising efficient modes of transportation for large, long-distance cargo (Ollila 1979). Several prototypes have been built to demonstrate technical feasibility and to develop economically practical systems. For example, the X-144 Aerofoil Boat designed by Lippisch (Koeivar 1977) skims over water and ground. The vehicle has a wing of chord 7 m and span 9 m, and it can cruise at an altitude of 1.5 m over water waves up to 1 m high.

Up to now, the extreme flat-ground-effect problem has only been studied by a few people. Strand, Roice & Fujita (1962) first indicated the 'channel-flow' character of the tightly constrained flow between the body and the ground. Widnall & Barrows (1970) gave the third-order analytical solution of the matched asymptotic expansions for a two-dimensional flat plate in the linear case. Regarding the nonlinear approach to the extreme flat-ground effect, the research work to date is limited to the first-order approximation. Tuck (1980) worked on the first-order boundary-value problem for a thin airfoil, using the method of matched asymptotic expansions. Newman (1982) gave the first-order analytical solution for a lifting surface of low aspect ratio.

And shortly after that, Tuck (1983) formulated the first-order boundary-value problem for the flow beneath a wing of arbitrary aspect ratio.

Our approach to the problem here has two differences to previous work. First, the previous investigations have been limited to the first-order gap flow approximation, except for the linear study of a two-dimensional flat plate. To consider the influence of the flow above the wing, a complete mathematical model should be formulated.

Secondly, the ground (or water surface) is actually curved in most of the practical problems related to ground effect. Sometimes, the amplitude of the ground undulation is comparable with the clearance beneath the wing, and so the curvature influence will be of the same order as the corresponding flat-ground effect, which will be studied in the present paper. In the complementary case, the curvature influence only causes a small perturbation of the flat-ground effect.

In this paper, the theory of a third-order matched asymptotic analysis is established for a three-dimensional thin wing moving uniformly in very close proximity to ground of curved surface in both the linear and nonlinear cases. It is shown that the flow field above the wing can be reduced to a direct problem, represented by a source–sink distribution on the upper surfaces of the wing and the wake and by a concentrated line source along the edges of the wing and the wake. The flow beneath the wing can be described by a set of linear two-dimensional elliptical partial differential equations. Based on this theory, some kinematic and dynamic equivalent relations between the curved-ground effect and its corresponding flat-ground effect in expansions up to third-order perturbations have been found.

2. General formulation of the boundary-value problem

A three-dimensional thin wing is set in motion with constant horizontal velocity U in very close proximity to a curved ground, as sketched in figure 1. If incompressible potential flow is assumed, the velocity potential $\phi(x,y,z,t)$ will satisfy the Laplace's equation with impermeable conditions on the wing surface $z=f_a(x,y,t)$ and the wavy ground surface $z=f_g(x-t,y)$, i.e.

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0, \tag{1a}$$

$$\phi_z=f_{at}+\phi_x f_{ax}+\phi_y f_{ay} \quad \text{on} \quad z=f_a(x,y,t), \tag{1b}$$

$$\phi_z=(\phi_x-1)f_{gx}+\phi_y f_{gy} \quad \text{on} \quad z=f_g(x-t,y), \tag{1c}$$

$$\phi=x, \quad \text{at infinity.} \tag{1d}$$

All quantities mentioned above and below have been normalized by c, U, and ρ, where c and ρ are the wing chord and fluid density respectively.

In addition, on the wake surface, which has shape $z=f_w(x,y,t)$ to be determined, the kinematic and dynamic boundary conditions should be imposed as follows:

$$\phi_z=f_{wt}+\phi_x f_{wx}+\phi_y f_{wy} \quad \text{on} \quad z=f_w(x,y,t), \tag{1e}$$

$$P(x,y,f_w^+,t)-P(x,y,f_w^-,t)=0, \tag{1f}$$

where superscripts + and − mean the upper and lower wake surfaces respectively. Equations (1a–f) comprise the present boundary-value problem.

After $\phi(x,y,z,t)$ is determined, the pressure distribution on the wing surface will be given by Bernoulli's equation

$$P=-[\phi_t+\tfrac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)-\tfrac{1}{2}]. \tag{2}$$

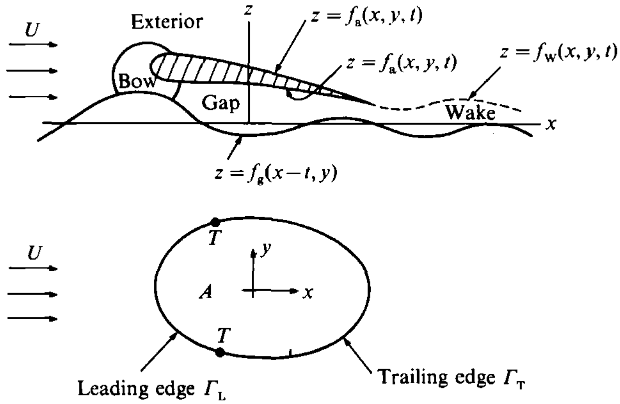

FIGURE 1. Sketch of a wing moving in close proximity to the curved ground, and the definition of the coordinate system and the four asymptotic regions.

Concerning the case of extreme ground effect, the following geometrical assumptions are made: (i) the gap clearance and the amplitude of the ground undulation have the same order of magnitude, of $O(\epsilon)$, i.e.

$$f_a, f_g, f_{gx}, f_{gy} = O(\epsilon); \tag{3a}$$

(ii) the variation of the wing elevation due to the combined effects of thickness, camber, angle of attack, and vertical motions is a small quantity of $O(\alpha)$, i.e.

$$f_{at}, f_{ax}, f_{ay} = O(\alpha). \tag{3b}$$

Two cases will be considered below: (i) the nonlinear case, $\alpha \sim \epsilon$; (ii) the linear case, $\alpha \ll \epsilon$.

For convenience when estimating the orders of magnitude of different quantities, we rewrite the equations of wing, ground, and wake surfaces as follows:

$$z = \epsilon + \alpha F_a(x, y, t), \quad z = \epsilon F_g(x-t, y), \quad z = \epsilon F_w(x, y, t),$$

where F_a, F_g, and F_w are functions of $O(1)$.

Compared with classical lifting-surface theory, the main difficulties of the present problem arise from two aspects, i.e. the multiple lengthscales in different parts of the flow field and the infinite boundary (the curved ground). In order to solve the problem by the method of matched asymptotic expansions, the flow field is divided into four separate asymptotic regions, as shown in figure 1, namely:

(i) Exterior region (E): the region above the wing and wake surfaces;

(ii) Gap region (G): the region beneath the lower surface of the wing;

(iii) Wake region (W): the region beneath the wake surface;

(iv) Bow region (B): any point in the region a distance of $O(\epsilon)$ from the wing's leading edge.

Here the leading edge means that part of the edge of the wing that does not shed a vortex sheet, the velocity being singular in its vicinity. The remaining part of the edge is referred to as the trailing edge. Because the velocity is finite near the trailing edge according to the Kutta condition, and the characteristic scale of the flow field is not changed across it either above or below the wing, there is no need to define the stern region at all. This is different from the previous papers mentioned in the Introduction, in which the stern region has been defined.

The mathematical solution to this problem simply requires expanding the solution in each region and matching them in the overlap parts. An appropriate sequence in which to conduct the matched asymptotic analysis in the various regions consists of the following four steps:

(i) the exterior solution is relatively simple and is found first;

(ii) the bow solution is deduced from the bow-region limit of the exterior solution;

(iii) the potential expansion in the gap region and the corresponding boundary-value problem at each order solution are obtained based on the above results;

(iv) the wake region is analysed in an analogous manner to the gap region.

3. Flow analysis by the method of asymptotic expansions

3.1. *Exterior region*

In this region, the asymptotic expansion of the velocity potential is written as

$$\phi^{\mathrm{E}} = x+\alpha\phi_1^{\mathrm{E}}(x,y,z,t)+o(\alpha), \tag{4}$$

and the boundary conditions on the wing and wake surfaces can be satisfied on the plane $z = 0$. Hence the boundary-value problem for ϕ_1^{E} is formulated as follows:

$$\phi_{1xx}^{\mathrm{E}}+\phi_{1yy}^{\mathrm{E}}+\phi_{1zz}^{\mathrm{E}} = 0, \tag{5a}$$

$$(\phi_1^{\mathrm{E}})_\infty = 0, \tag{5b}$$

$$\phi_{1z}^{\mathrm{E}}\big|_{z=0} = \begin{cases} w(x,y,t) & \text{on } \Sigma \\ 0 & \text{elsewhere,} \end{cases} \tag{5c}$$

where Σ denotes the wing plane A plus the wake plane W. The upwash velocity w takes the form

$$w(x,y,t) = \begin{cases} F_{\mathrm{a}t}^{+}+F_{\mathrm{a}x}^{+} & \text{on } A \\ F_{\mathrm{w}t}+F_{\mathrm{w}x} & \text{on } W, \end{cases} \tag{5d}$$

where the superscript + means the upper wing surface.

Based on (5*a–d*) and the principle of minimum singularity (Van Dyke 1964), ϕ_1^{E} can be determined by a source–sink distribution on the upper surfaces of the wing and wake and a concentrated line source along the leading edge Γ_{L} of the wing and the sides ν of the wake, i.e.

$$\phi_1^{\mathrm{E}} = -\frac{1}{4\pi}\iint_\Sigma \frac{w(x_1,y_1,t)\,\mathrm{d}x_1\,\mathrm{d}y_1}{[(x_1-x)^2+(y_1-y)^2+z^2]^{\frac{1}{2}}}+\frac{1}{2\pi}\int_{\Gamma_{\mathrm{L}}+\nu}\frac{\sigma(s_1,t)\,\mathrm{d}s_1}{[(x_1-x)^2+(y_1-y)^2+z^2]^{\frac{1}{2}}}, \tag{6}$$

where the unknown line source $\sigma(s_1,t)$ and the surface source $w(x_1,y_1,t)$ on the wake will be determined by matching ϕ_1^{E} with the first-order solutions of the gap region and the wake region.

Now we examine the asymptotic behaviour of ϕ_1^{E} in the vicinity of the leading edge. Let us consider an arbitrary point $P(s)$ on the leading edge and set a local Cartesian coordinate system (n,τ,z) with its origin at $P(s)$ and with the n- and τ-axes normal and tangent to Γ_{L} at $P(s)$ respectively in the horizontal plane, as sketched in figure 2. Thus ϕ_1^{E} on the $\tau = 0$ plane can be written as

$$\phi_1^{\mathrm{E}} = -\frac{1}{4\pi}\int_\Sigma \frac{G(n_1,\tau_1,t;s)\,\mathrm{d}n_1\,\mathrm{d}\tau_1}{[(n_1-n)^2+\tau_1^2+z^2]^{\frac{1}{2}}}+\frac{1}{2\pi}\int_{\Gamma_{\mathrm{L}}+\nu}\frac{\sigma(s_1,t)\,\mathrm{d}s_1}{[(n_1-n)^2+\tau_1^2+z^2]^{\frac{1}{2}}}, \tag{7}$$

where $G(n_1,\tau_1,t;s) = w(x(n_1,\tau_1,s),y(n_1,\tau_1,s),t)$, and n_1 and τ_1 are the coordinates of the point s_1 in the second integral.

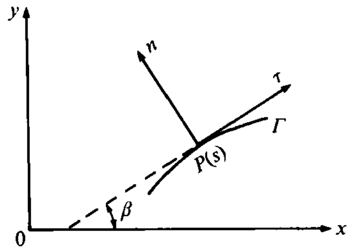

FIGURE 2. Definition of the local Cartesian coordinates (n, τ, z).

The scales of bow region are $O(\epsilon)$ in the n- and z-directions and $O(1)$ in the τ-direction, while the curvature of the leading edge is assumed to be of $O(1)$. Hence the local coordinates n, z should be stretched into

$$N = n/\epsilon, \quad Z = z/\epsilon. \tag{8}$$

The inner limit of ϕ_1^{E}, with N, Z fixed as $\epsilon \to 0$, should be calculated to match with the bow-region solution. In doing so, the self-induced potential in the neighbourhood of the line source in (7) must be treated carefully. Here it is analysed based on the method which Batchelor (1967) used to treat the similar problem for a vortex line.

First, the second integral in (7) is divided:

$$\int_{\Gamma_{\mathrm{L}}+\nu} \ldots \mathrm{d}s_1 = \int_{-L+s}^{L+s} \frac{\sigma(s_1,t)\,\mathrm{d}s_1}{[(n_1-n)^2+\tau_1^2+z^2]^{\frac{1}{2}}} + \left[\int_{\Gamma_{\mathrm{L}}+\nu} - \int_{-L+s}^{L+s}\right] \frac{\sigma(s_1,t)\,\mathrm{d}s_1}{[(n_1-n)^2+\tau_1^2+z^2]^{\frac{1}{2}}}, \tag{9}$$

where L is a constant of $O(1)$. Thus the first integral is singular and the quantity in the brackets is not.

Next, noticing that as $l = s_1 - s \to 0$ in the contour Γ_{L}, $\tau_1 = l + O(l^5)$, $n_1 = -\frac{1}{2}bl^2 + O(l^3)$, where b is the curvature of Γ_{L}, we have

$$\frac{1}{[(n_1-n)^2+\tau_1^2+z^2]^{\frac{1}{2}}} = \frac{1}{[n^2+z^2+l^2(1+bn)]^{\frac{1}{2}}} + O(l,n)\left[\frac{l}{[n^2+z^2+l^2(1+bn)]^{\frac{1}{2}}}\right]^3 \quad \text{as} \quad l \to 0.$$

Let us rewrite the integrand of the first integral in (9) as follows:

$$\frac{\sigma(s_1,t)}{[(n_1-n)^2+\tau_1^2+z^2]^{\frac{1}{2}}} = \frac{\sigma(s,t)}{[n^2+z^2+l^2(1+bn)]^{\frac{1}{2}}} + \frac{\sigma(s_1,t)-\sigma(s,t)}{[(n_1-n)^2+\tau_1^2+z^2]^{\frac{1}{2}}}$$
$$+\sigma(s,t)\left\{\frac{1}{[(n_1-n)^2+\tau_1^2+z^2]^{\frac{1}{2}}} - \frac{1}{[n^2+z^2+l^2(1+bn)]^{\frac{1}{2}}}\right\}, \tag{10}$$

with the first singular term being integrable and the remaining terms being finite and continuous.

And then, substituting (10) into (9), we obtain the inner limit of ϕ_1^{E}:

$$(\phi_1^{\mathrm{E}})^{\mathrm{B}} = -\frac{\sigma(s,t)}{\pi} \ln\left[\epsilon(N^2+Z^2)^{\frac{1}{2}}\right] + c(s,t), \tag{11a}$$

where

$$c(s,t) = -\frac{1}{4\pi}\iint_{\Sigma} \frac{G(n_1,\tau_1,t;s)\,\mathrm{d}n_1\,\mathrm{d}\tau_1}{(n_1^2+\tau_1^2)^{\frac{1}{2}}} + \frac{1}{2\pi}\left\{\int_{-L}^{L} \frac{\sigma(s_1,t)-\sigma(s,t)}{(n_1^2+\tau_1^2)^{\frac{1}{2}}}\,\mathrm{d}l\right.$$
$$\left. +\sigma(s,t)\int_{-L}^{L}\left[\frac{1}{(n_1^2+\tau_1^2)^{\frac{1}{2}}} - \frac{1}{|l|}\right]\mathrm{d}l + 2\sigma(s,t)\ln(2L) + \left[\int_{\Gamma_{\mathrm{L}}+\nu} - \int_{-L+s}^{L+s}\right]\frac{\sigma(s_1,t)}{(n_1^2+\tau_1^2)^{\frac{1}{2}}}\,\mathrm{d}s_1\right\}. \tag{11b}$$

The inner limit of ϕ_0^E on the $\tau = 0$ plane is the same as the free-stream solution. In local coordinates this is written

$$(\phi_0^E)^B = x|_{\Gamma_L} - \epsilon N \sin\beta, \tag{12}$$

where β is the angle between Γ_L and the x-axis.

3.2. *Bow region*

Noting the characteristic scales of the bow region mentioned in §3.1 and using the solutions of $(\phi_0^E)^B$ and $(\phi_1^E)^B$ given in (12) and (11), we can write the velocity potential in the bow region in the following form:

$$\phi^B = x|_{\Gamma_L} - \epsilon N \sin\beta + \frac{\alpha}{\epsilon}\phi_0^B(N,s,Z,t) + \alpha \ln\epsilon\, \phi_1^B(N,s,Z,t) + \alpha\phi_2^B(N,s,Z,t) + o(\alpha), \tag{13}$$

where ϕ_k^B $(k = 0,1,2)$ should satisfy the two-dimensional Laplace's equation, i.e.

$$\left[\frac{\partial^2}{\partial N^2} + \frac{\partial^2}{\partial Z^2}\right]\phi_k^B = 0 \quad \text{for} \quad k = 0,1,2. \tag{14a}$$

The asymptotic limits of ϕ^B matched with neighbouring flow regions will be determined in order.

From (12) and (11), we have the limits matched with the exterior region:

$$\left.\begin{aligned} (\phi_0^B)^E &= 0, \\ (\phi_1^B)^E &= -\frac{\sigma(s,t)}{\pi}, \\ (\phi_2^B)^E &= -\frac{\sigma(s,t)}{\pi}\ln(N^2+Z^2)^{\frac{1}{2}} + c(s,t). \end{aligned}\right\} \tag{14b}$$

In the gap region we can get from (1*b*)

$$\phi_z^G = O(\alpha),$$

so that

$$\phi^G(x,y,z,t) = \phi^G(x,y,0,t) + O(\alpha\epsilon) \to \phi^G|_{\Gamma_L} + \epsilon N \phi_n^G|_{\Gamma_L} \quad \text{as} \quad n \to 0. \tag{15}$$

Comparing (15) and (13), we have

$$\phi^G = x + \frac{\alpha}{\epsilon}\phi_0^G(x,y,t) + \alpha\ln\epsilon\,\phi_1^G(x,y,t) + \alpha\phi_2^G(x,y,t) + o(\alpha). \tag{16}$$

Hence the asymptotic limits of ϕ_k^B $(k = 0,1,2)$ matched with the gap region are

$$\left.\begin{aligned} (\phi_k^B)^G &= \phi_k^G|_{\Gamma_L} \quad \text{for} \quad k = 0,1, \\ (\phi_2^B)^G &= \phi_2^G|_{\Gamma_L} + N\phi_{0n}^G|_{\Gamma_L}. \end{aligned}\right\} \tag{17a}$$

The boundary conditions of the bow region on the surfaces of the wing and ground can be obtained from (1*b*), (1*c*) respectively, i.e.

$$\left.\begin{aligned} \phi_{kZ}^B &= 0 \quad \text{for} \quad k = 0,1 \\ \phi_{2Z}^B &= \bar{F}_{an}(N\phi_{0N}^B)_N \end{aligned}\right\} \quad \text{on} \quad Z = \bar{F}_a, \quad N < 0, \tag{17b}$$

and

$$\left.\begin{aligned} \phi_{kZ}^B &= 0 \quad \text{for} \quad k = 0,1 \\ \phi_{2Z}^B &= \bar{F}_{gn}(N\phi_{0N}^B)_N \end{aligned}\right\} \quad \text{on} \quad Z = \bar{F}_g, \tag{17c}$$

where $\bar{F}_a = F_a|_{\Gamma_L}, \bar{F}_g = F_g|_{\Gamma_L}$ and $\bar{F}_{an} = F_{an}|_{\Gamma_L}, \bar{F}_{gn} = F_{gn}|_{\Gamma_L}$.

Now we are ready to solve the boundary-value problem (14*a*, *b*), (17*a*–*c*). It can be obtained that

$$\phi_0^{\mathrm{B}} = 0, \quad \phi_1^{\mathrm{B}} = -\frac{\sigma(s,t)}{\pi}. \tag{18}$$

Substituting (18) into (17*b*) and (17*c*), we have

$$\left.\begin{aligned} \phi_{2Z}^{\mathrm{B}} &= 0 \quad \text{on} \quad Z = \bar{F}_a, \quad N < 0, \\ \phi_{2Z}^{\mathrm{B}} &= 0 \quad \text{on} \quad Z = \bar{F}_g. \end{aligned}\right\} \tag{19}$$

Thus the solution of ϕ_2^{B} takes the form

$$\phi_2^{\mathrm{B}} = \phi_{\mathrm{B}} + c_1(s,t),$$

where ϕ_{B} is the solution of Laplace equation (14*a*), satisfying the homogeneous boundary conditions (19). The corresponding complex potential Φ_{B} can be solved by Schwarz–Christoffel conformal mapping (Widnall & Barrows 1970) in the implicit form:

$$\frac{\Phi_{\mathrm{B}}}{Q} + \frac{1}{\pi}\left[1 + \exp\left(\frac{\pi \Phi_{\mathrm{B}}}{Q}\right)\right] = \frac{N + \mathrm{i}(Z - \bar{F}_g)}{\bar{H}_a}, \tag{20}$$

where $\bar{H}_a = \bar{F}_a - \bar{F}_g$, and Q is a coefficient to be determined. Thus the asymptotic expressions for ϕ_2^{B} are

$$\left.\begin{aligned} (\phi_2^{\mathrm{B}})^{\mathrm{E}} &= \frac{Q}{\pi}\ln\left(\frac{\pi}{\bar{H}_a}(N^2+Z^2)^{\frac{1}{2}}\right) + c_1(s,t), \\ (\phi_2^{\mathrm{B}})^{\mathrm{G}} &= Q\left(\frac{N}{\bar{H}_a} - \frac{1}{\pi}\right) + c_1(s,t). \end{aligned}\right\} \tag{21a}$$

Comparing (21*a*) with (14*b*) and (17*a*), we get the coefficients Q and c:

$$\left.\begin{aligned} Q &= -\sigma(s,t) = \bar{H}_a \phi_{0n}^{\mathrm{G}}|_{\Gamma_L}, \\ c_1(s,t) &= c(s,t) - \frac{\bar{H}_a}{\pi}\ln\left[\frac{\pi}{\bar{H}_a}\right]\phi_{0n}^{\mathrm{G}}|_{\Gamma_L}. \end{aligned}\right\} \tag{21b}$$

Finally, the boundary conditions for ϕ_k^{G} $(k = 0, 1, 2)$ along the leading edge are obtained from (17*a*), i.e.

$$\left.\begin{aligned} \phi_0^{\mathrm{G}}|_{\Gamma_L} &= 0, \\ \phi_1^{\mathrm{G}}|_{\Gamma_L} &= \frac{\bar{H}_a}{\pi}\phi_{0n}^{\mathrm{G}}|_{\Gamma_L}, \\ \phi_2^{\mathrm{G}}|_{\Gamma_L} &= c(s,t) - \frac{\bar{H}_a}{\pi}\ln\left[\frac{\pi e}{\bar{H}_a}\right]\phi_{0n}^{\mathrm{G}}|_{\Gamma_L}. \end{aligned}\right\} \tag{22}$$

3.3. *Gap region*

In the gap region, $x, y = O(1)$, $z = O(\epsilon)$, and from (1*b*), we have

$$\phi_z^{\mathrm{G}} = O(\alpha).$$

Hence we have obtained the expansions of ϕ^{G}, as shown in (16).

Next, we take a control volume element in the gap region in the range $x \leqslant \xi \leqslant x+\mathrm{d}x$, $y \leqslant \eta \leqslant y+\mathrm{d}y$, and $f_g \leqslant z \leqslant f_a^-$. Then the equation of conservation of mass is written as

$$\frac{\partial}{\partial x}\int_{f_g}^{f_a^-} \phi_x^G \,\mathrm{d}z + \frac{\partial}{\partial y}\int_{f_g}^{f_a^-} \phi_y^G \,\mathrm{d}z + \frac{\partial}{\partial t}(f_a^- - f_g) = 0. \tag{23}$$

Substituting (15) into (23), we get

$$H_{at} + (H_a \phi_x^G)_x + (H_a \phi_y^G)_y = O(\alpha\epsilon), \tag{24}$$

where $H_a = F_a^- - F_g$. Again, ($1f$) has been given as the Kutta condition at the trailing edge Γ_T, which can be simplified in the form

$$\phi_t^G + \tfrac{1}{2}[(\phi_x^G)^2 + (\phi_y^G)^2] = \tfrac{1}{2} + \alpha(\phi_{1x}^E + \phi_{1t}^E) + O(\alpha\epsilon) \quad \text{on} \quad \Gamma_T. \tag{25}$$

Substituting (16) into (24), (25), and equating terms of the same order, we can formulate the boundary-value problem for each order of the potential $\phi_k^G (k = 0, 1, 2)$, which is again complemented with the leading-edge boundary conditions (22).

For the first-order solution $\phi_0^G(x, y, t)$, we have

$$F_{at}^- + F_{ax}^- + (H_a \phi_{0x}^G)_x + (H_a \phi_{0y}^G)_y = 0 \quad \text{on} \quad A, \tag{26a}$$

$$\phi_0^G = 0 \quad \text{on} \quad \Gamma_L, \tag{26b}$$

$$\phi_{0t}^G + \phi_{0x}^G + \tfrac{1}{2}[(\phi_{0x}^G)^2 + (\phi_{0y}^G)^2] = 0 \quad \text{on} \quad \Gamma_T. \tag{26c}$$

For the second-order solution $\phi_1^G(x, y, t)$, we have

$$(H_a \phi_{1x}^G)_x + (H_a \phi_{1y}^G)_y = 0 \quad \text{on} \quad A, \tag{27a}$$

$$\phi_1^G = \frac{\bar{H}_a}{\pi}\phi_{0n}^G \quad \text{on} \quad \Gamma_L, \tag{27b}$$

$$\phi_{1t}^G + \phi_{1x}^G + \phi_{0x}^G \phi_{1x}^G + \phi_{0y}^G \phi_{1y}^G = 0 \quad \text{on} \quad \Gamma_T. \tag{27c}$$

For the third-order solution $\phi_2^G(x, y, t)$, we have

$$(H_a \phi_{2x}^G)_x + (H_a \phi_{2y}^G)_y = 0 \quad \text{on} \quad A, \tag{28a}$$

$$\phi_2^G = c(s, t) - \frac{\bar{H}_a}{\pi}\ln\left[\frac{\pi e}{\bar{H}_a}\right]\phi_{0n}^G \quad \text{on} \quad \Gamma_L, \tag{28b}$$

$$\phi_{2t}^G + \phi_{2x}^G + \phi_{0x}^G \phi_{2x}^G + \phi_{0y}^G \phi_{2y}^G = \phi_{1x}^E + \phi_{1t}^E \quad \text{on} \quad \Gamma_T. \tag{28c}$$

Thus the gap flow problem is reduced to a series of two-dimensional linear elliptical equations, in which the gap clearance appears as a variable coefficient, subjected to nonlinear mixed boundary conditions. Physically, the equation governing ϕ_0^G can be interpreted as the conservation of mass in the two-dimensional channel beneath the wing with known gap clearance and added mass. The equations governing ϕ_1^G and ϕ_2^G come from the conservation of mass in the same two-dimensional channel with no added mass.

In classical wing theory, the wake generally sheds from the part of the wing edge after the wing tips, except for a highly unsteady wing (cf. Newman & Wu 1973). Nevertheless, for a wing in nonlinear extreme ground effect, the lateral deflection of streamlines in the gap region is comparable with the span, and the transition points between the leading edge Γ_L and the trailing edge Γ_T are usually not at the wing tips. They should be determined as part of the solution. Newman (1982) found the transition point locations, where the mean velocity vectors (above and below the wing) are tangent to the edge contour for a low-aspect-ratio wing in extreme flat-ground effect. Tuck (1983) extended this conclusion to arbitrary wings in steady flow. Here we will show that such a conclusion remains valid for unsteady wings in extreme curved-ground effect as well. In fact, from (26*b*) we know

$$\phi^{G}_{0\tau} = 0, \quad \phi^{G}_{0t} = 0 \quad \text{on} \quad \Gamma_{L}. \tag{29a}$$

Similarly (26*c*) can be written as

$$\phi^{G}_{0t} - \sin\beta\phi^{G}_{0n} + \cos\beta\phi^{G}_{0\tau} + \tfrac{1}{2}[(\phi^{G}_{0n})^2 + (\phi^{G}_{0\tau})^2] = 0 \quad \text{on} \quad \Gamma_{T}. \tag{29b}$$

If we demand that the tangential velocity $\phi^{G}_{0\tau}$ be continuous across the transition point, both the leading- and trailing-edge conditions (29*a*) and (29*b*) should be satisfied simultaneously, so that $\phi^{G}_{0n} = 2\sin\beta$. Hence the mean normal velocity components at the transition points are zero, as the normal components of the velocity above the wing are $-\sin\beta$. The higher-order correction of the transition position is of less importance, because it does not affect the solutions up to third order.

Now we consider the linear problem, in which case $\alpha \ll \epsilon$. The boundary-value problem (26), (27) and (28) can be reduced to the following forms:

For the first-order solution $\phi^{G}_{0}(x,y,t)$, we have

$$\left.\begin{aligned} &F^{-}_{at} + F^{-}_{ax} + [(1-F_g)\phi^{G}_{0x}]_x + [(1-F_g)\phi^{G}_{0y}]_y = 0 && \text{on} \quad A, \\ &\phi^{G}_{0} = 0 && \text{on} \quad \Gamma_{L}, \\ &\phi^{G}_{0t} + \phi^{G}_{0x} = 0 && \text{on} \quad \Gamma_{T}. \end{aligned}\right\} \tag{30}$$

For the second-order solution $\phi^{G}_{1}(x,y,t)$, we have

$$\left.\begin{aligned} &[(1-F_g)\phi^{G}_{1x}]_x + [(1-F_g)\phi^{G}_{1y}]_y = 0 && \text{on} \quad A, \\ &\phi^{G}_{1} = (1-F_g)\phi^{G}_{0n}/\pi && \text{on} \quad \Gamma_{L}, \\ &\phi^{G}_{1t} + \phi^{G}_{1x} = 0 && \text{on} \quad \Gamma_{T}. \end{aligned}\right\} \tag{31}$$

For the third-order solution $\phi^{G}_{2}(x,y,t)$, we have

$$\left.\begin{aligned} &[(1-F_g)\phi^{G}_{2x}]_x + [(1-F_g)\phi^{G}_{2y}]_y = 0 && \text{on} \quad A, \\ &\phi^{G}_{2} = c(s,t) - \frac{1-F_g}{\pi}\ln\left[\frac{\pi e}{1-F_g}\right]\phi^{G}_{0n} && \text{on} \quad \Gamma_{L}, \\ &\phi^{G}_{2t} + \phi^{G}_{2x} = \phi^{E}_{1x} + \phi^{E}_{1t} && \text{on} \quad \Gamma_{T}. \end{aligned}\right\} \tag{32}$$

One can find that the first-order transition points are located at the wing tips in the linear problem.

3.4. *Wake region*

When a wing moves through an infinite fluid, the vortex sheet shed from the wing edge often rolls up (Batchelor 1967). This is partly because the velocity components $\phi_n - \sin\beta$ and ϕ_z of the flow that are induced are of the same order, $O(\alpha)$.

For a wing in extreme curved-ground effect, on the other hand, the normal velocity component $\frac{1}{2}\phi^{G}_{0n}$ of $O(\alpha/\epsilon)$ of the mean flow induced above and below the wing is much larger than the vertical velocity of ϕ_z, which is of $O(\alpha)$. Also, the flow mean is outward from the wing planform A in the vicinity of the trailing edge, which carries the vortex sheet so that it moves predominantly in the horizontal direction. The vortex sheet will assume a weakly curved surface close to the curved ground.

Thus, there is no essential mathematical difference between the wake region and gap region, except for the fact that the boundary surface of the wake region $z = \epsilon F_W(x,y,t)$ is to be solved. The approximate potential equation in the wake region can be obtained as in (24), (25), namely

$$H_t^W + (H^W \phi_x^W)_x + (H^W \phi_y^W)_y = O(\alpha\epsilon), \tag{33a}$$

$$\phi_t^W + \tfrac{1}{2}[(\phi_x^W)^2 + (\phi_y^W)^2] = \tfrac{1}{2} + \alpha(\phi_{1x}^E + \phi_{1t}^E)|_{z=0} + O(\alpha\epsilon), \tag{33b}$$

where $H^W = F_W - F_g$ is the clearance between the wake surface and the ground. In addition, the continuity conditions should be imposed across the trailing edge Γ_T:

$$H^W = H_a \quad \text{on} \quad \Gamma_T, \tag{33c}$$

$$\phi^W = \phi^G, \quad \phi_n^W = \phi_n^G \quad \text{on} \quad \Gamma_T. \tag{33d}$$

The asymptotic expansions of ϕ^W and H^W are

$$\phi^W = x + \frac{\alpha}{\epsilon}\phi_0^W(x,y,t) + \alpha \ln \epsilon \phi_1^W(x,y,t) + \alpha\phi_2^W(x,y,t) + o(\alpha), \tag{34a}$$

$$H^W = \bar{H}_a + \frac{\alpha}{\epsilon}H_0^W(x,y,t) + \alpha \ln \epsilon H_1^W(x,y,t) + \alpha H_2^W(x,y,t) + o(\alpha). \tag{34b}$$

Substituting them into (33), we can formulate the following boundary-value problems.

For the first-order solution $\phi_0^W(x,y,t)$ and $H_0^W(x,y,t)$, we have

$$\left.\begin{array}{lll} H_{0t}^W + H_{0x}^W + (H_0^W \phi_{0x}^W)_x + (H_0^W \phi_{0y}^W)_y = 0 & \text{on} & W, \\ \phi_{0t}^W + \phi_{0x}^W + \tfrac{1}{2}[(\phi_{0x}^W)^2 + (\phi_{0y}^W)^2] = 0 & \text{on} & W, \\ \phi_0^W = 0 & \text{on} & \nu, \\ H_0^W = 0, \quad \phi_0^W = \phi_0^G, \quad \phi_{0n}^W = \phi_{0n}^G & \text{on} & \Gamma_T. \end{array}\right\} \tag{35}$$

For the second-order solution $\phi_1^W(x,y,t)$ and $H_1^W(x,y,t)$, we have

$$\left.\begin{array}{lll} H_{1t}^W + H_{1x}^W + (H_0^W \phi_{1x}^W)_x + (H_0^W \phi_{1y}^W)_y + (H_1^W \phi_{0x}^W)_x + (H_1^W \phi_{0y}^W)_y = 0 & \text{on} & W, \\ \phi_{1t}^W + \phi_{1x}^W + \phi_{0x}^W \phi_{1x}^W + \phi_{0y}^W \phi_{1y}^W = 0 & \text{on} & W, \\ \phi^W = \dfrac{H_0}{\pi}\phi_{0n}^W & \text{on} & \nu, \\ H_1^W = 0, \quad \phi_1^W = \phi_1^G, \quad \phi_{1n}^W = \phi_{1n}^G & \text{on} & \Gamma_T. \end{array}\right\} \tag{36}$$

For the third-order solution $\phi_2^W(x,y,t)$ and $H_2^W(x,y,t)$, we have

$$\left.\begin{array}{lll} H_{2t}^W + H_{2x}^W + (H_0^W \phi_{2x}^W)_x + (H_0^W \phi_{2y}^W)_y + (H_2^W \phi_{0x}^W)_x + (H_2^W \phi_{0y}^W)_y = 0 & \text{on} & W, \\ \phi_{2t}^W + \phi_{2x}^W + \phi_{0x}^W \phi_{2x}^W + \phi_{0y}^W \phi_{2y}^W = \phi_{1x}^E + \phi_{1t}^E & \text{on} & W, \\ \phi_2^W = c(s,t) - \dfrac{H_0}{\pi}\ln\left[\dfrac{\pi e}{H_0}\right]\phi_{0n}^W & \text{on} & \nu, \\ H_2^W = 0, \quad \phi_2^W = \phi_2^G, \quad \phi_{2n}^W = \phi_{2n}^G & \text{on} & \Gamma_T, \end{array}\right\} \tag{37}$$

where $H_0 = \bar{H}_a + \alpha/\epsilon H_0^W(x,y,t)$, and the boundary conditions of ϕ_k^W $(k = 0, 1, 2)$ at the sides ν of the wake are deduced in the same way as those for the leading edge of the wing in the gap region. The condition for determining the position of ν is that the mean (above and below the wake) velocity vector on ν is tangent to that curve.

These boundary-value problems can also be simplified for the linear problem in the same way as in the gap region. In particular, one can find that the first-order wake surface is

$$z = \epsilon + \epsilon F_g(x-t, y),$$

with its sides being lines starting from the wing tips and parallel to the flow direction at infinity.

4. The equivalence between the extreme curved-ground effect and the corresponding flat-ground effect

Now some kinematic and dynamic equivalent relations between the extreme curved-ground effect and the corresponding flat-ground effect can be found, which are valid in both linear and nonlinear cases. To this end, two relevant problems are compared:

(i) A thin wing $z = f_a(x, y, t)$, (x, y) on A, moves with the velocity $\boldsymbol{v} = -\boldsymbol{i}$ in very close proximity to the curved ground surface $z = f_g(x-t, y)$;

(ii) An equivalent wing $z = f_a(x, y, t) - f_g(x-t, y)$, (x, y) on A, moves with the same velocity in very close proximity to the flat ground surface $z = 0$. That is, the under surface of the equivalent wing undulates in such a way as to match with ground curvature of the problem (i), which is moving past at the free-stream velocity.

Thus we can see that, for the above-mentioned two relevant problems, ϕ_0^G, ϕ_0^W, and H_0^W are the same according to (26) and (35) respectively. From (6) and (21), we still know that solutions in the exterior region $\phi^E = x + \alpha\phi_1^E$ are equal to each other. Similarly, the following terms up to second-order remain unchanged in the bow region. Equations (27) and (28) state that the gap solutions up to third-order $\phi^G = x + \alpha/\epsilon\phi_0^G + \alpha \ln \epsilon\phi_1^G + \alpha\phi_2^G$ are same. Further, (36) and (37) state that the same results hold for the wake-region solutions $\phi^W = x + \alpha/\epsilon\phi_0^W + \alpha \ln \epsilon\phi_1^W + \alpha\phi_2^W$, and for the clearance $H^W = \bar{H}_a + \alpha/\epsilon H_0^W + \alpha \ln \epsilon H_1^W + \alpha H_2^W$. These results have been identified as the kinematic equivalence between the extreme curved-ground effect and the corresponding flat-ground effect.

In addition, the dynamic equivalence between those two problems can be deduced from Bernoulli's equation. First, it is easy to see that the added pressures over the value at infinity are same up to third-order (i.e. $P = \alpha/\epsilon P_0 + \alpha \ln \epsilon P_1 + \alpha P_2$), except for the bow region. In the bow region, the added pressure can be expressed as

$$P = -\frac{1}{2}\frac{\alpha^2}{\epsilon^2}[(\phi_{2N}^B)^2 + (\phi_{2Z}^B)^2] + o(\alpha). \tag{38}$$

One can see from the solution (20) of ϕ_2^B that the first-order pressures on the surface of the wing are equal in this region. But, from the results of the pressure integration over the wing surface, the total aerodynamic forces (lift, pressure drag and pitching moment, etc.) again prove to be the same up to third-order terms for both cases, where the lift and pitching moment take the form

$$F = \frac{\alpha}{\epsilon}F_0 + \alpha \ln \epsilon F_1 + \alpha F_2 \tag{39}$$

and the pressure drag takes the form

$$D = \frac{\alpha^2}{\epsilon}D_0 + \alpha^2 \ln \epsilon D_1 + \alpha^2 D_2. \tag{40}$$

Based on the analysis in §3, one can see that the first- and second-order aerodynamic forces F_0, D_0 and F_1, D_1 are dependent only on the first- and second-order gap solutions ϕ_0^G and ϕ_1^G respectively. However, to obtain the third-order aerodynamic forces F_2 and D_2, one must obtain the solutions of ϕ_0^W, H_0^W, ϕ_1^E, and ϕ_2^G in a matching procedure.

Incidentally, it is known from the dynamic equivalence that the dynamic effect of the ground curvature is only dependent on the ground elevation beneath the wing to the third-order approximation.

5. Summary and conclusions

The nonlinear problem for a thin wing moving uniformly in very close proximity to curved ground has been formulated and analysed by using the method of matched asymptotic expansions. Summarizing the analyses, the following conclusions can be made.

The downwash of $O(\alpha)$ from the under surface of the wing induces first-order two-dimensional channel flow ϕ_0^G with a horizontal velocity of $O(\alpha/\epsilon)$ in the gap region beneath the wing, which can be described by a linear two-dimensional elliptic partial differential equation. Solutions are required subject to separate leading-edge and (nonlinear) trailing-edge boundary conditions, with the transition point occurring where the mean (above and below the wing) velocity vector is tangent to the wing edge. These results were given by Tuck (1983) for a steady wing near a plane wall using intuitive arguments, and are now extended to an unsteady wing near curved ground in a systematic asymptotic procedure. It is also found that the influence of the curvature of the ground is of the same order as the corresponding plane-ground effect; in particular, it is of $O(1)$ for the nonlinear problems.

The wake surface assumes a vortex-sheet structure with an elevation of $O(\epsilon)$ close to the curved ground, rather than the vortex rolls that appear in conventional wing theory. The first-order wake flow ϕ_0^W beneath the wake is also two-dimensional channel flow with the horizontal velocity of $O(\alpha/\epsilon)$, except that the boundary surface of this region is a free surface similar to the water surface in shallow-water wave theory. The condition for determining the position of the sides ν of the wake is that the mean (above and below the wake) velocity vector on ν is tangent to that curve. For the linear problems, the first-order wake surface undulates in such a way as to match with the ground curvature, with its sides being two lines starting from the wing tips and parallel to the flow direction at infinity.

The exterior flow above the wing is induced by the upwashes of $O(\alpha)$ from the upper surfaces of the wing and wake. In addition, it is induced by the 'crack-inflow' (or 'crack-outflow') with a mass flux of $O(\alpha)$ around the leading edge Γ_L of the wing and the sides ν of the wake, due to the first-order channel flows with horizontal velocity of $O(\alpha/\epsilon)$ in the gap region and wake region. Therefore, in an analogous way to Windnall & Barrows' (1970) linearized two-dimensional study, the disturbance potential ϕ_1^E of $O(\alpha)$ in the exterior region has been expressed by a source–sink distribution on the upper surfaces of the wing and wake, and the concentrated line source around the leading edge of the wing and the sides of the wake, with the strength of the singularities known or given in terms of the first-order gap solution ϕ_0^G and first-order wake solutions ϕ_0^W, H_0^W.

The disturbance potential ϕ_1^E of $O(\alpha)$ in the exterior region will affect the 'crack-inflow' (or 'crack-outflow') around the leading edge Γ_L of the wing and the sides ν of the wake, and the pressure distribution on the wake surface. As a result, the second-order flows ϕ_1^G, ϕ_1^W of $O(\alpha \ln \epsilon)$ and third-order flows ϕ_2^G and ϕ_2^W of $O(\alpha)$ are induced in the gap region and wake region, which are two-dimensional channel flows which can be described by linear partial differential equations with two independent variables.

As for the dynamic problem of the wing, the matching is unnecessary up to the second-order approximation. In fact, to obtain the second-order aerodynamic forces, one just needs the gap solutions ϕ_0^G and ϕ_1^G. However, to obtain the third-order aerodynamic forces, one must also compute the solutions of ϕ_0^W, H_0^W, ϕ_1^E, and ϕ_3^G in a matching procedure. To a first-order approximation, the lift is proportional to α/ϵ, and the ratio of lift to pressure drag is proportional to α. In other words, for a fixed α, the smaller the clearance is the larger the lift will become; and as the ratio of α/ϵ is fixed to meet a certain lift demand, the pressure drag will be decreased with the clearance. Thus, when a wing is designed to take advantage of the ground effect, the greatest interest lies in very close proximity.

Finally, the important result given in the present paper is that the unsteady flow problem for a thin wing in extreme curved-ground effect can be reduced to an equivalent one in flat-ground effect. Some kinematic and dynamic equivalent relations between them have been established.

The author wishes to express sincere thanks to Professor Bing-Gang Tong and Professor Li-Xian Zhuang for many stimulating discussions concerning this work, and his appreciation to Professor T. Y. Wu of CALTECH for his valuable suggestions about this topic. The author is also grateful to the referees of this paper for their critical review of the manuscript.

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